A DIVE BACK INTO DIARIES

Souradip Das

April 13, 2023

Session 10

About Myself: Hello! I am Souradip Das, a student of class 10. I live in the city Kolkata. I had taken my first Math Circle Session with 6 students from a rural school in The Sunderbans. We had a lot of fun and had lots of discussions. This diary will sum up all the discussions we had in the 10th session.

1 Introduction

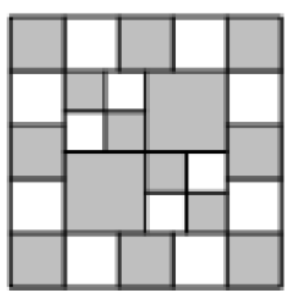
Note:- Due to my Board Exams this year, I was 4 months busy for its preparation so I couldn't get time to start writing diaries, although I did take math circles sessions fairly regularly. Thus I wish to start my 10th record with this session.

We had 1 student in the session, Raghav. He was a 7th grader.

We had discussed a total of 7 problems, 5 of which were given by cheenta, and the other 2 were made by me. I hope this Session will cover all the discussions that we had together.

2 Question 1

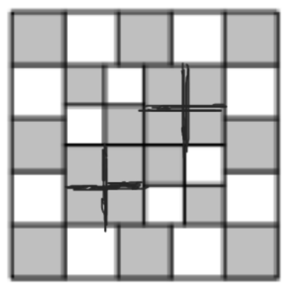
We started with the 1st Question.
The diagram shows a large square divided into squares of three diferent sizes. What percentage of the large square is shaded?



This problem was fairly easy for a 7th grader, so I gave some time to Raghav to solve it. After some time, Raghav told me the answer was 68%. Since I hadn't solved it yet, from here we started the discussion.

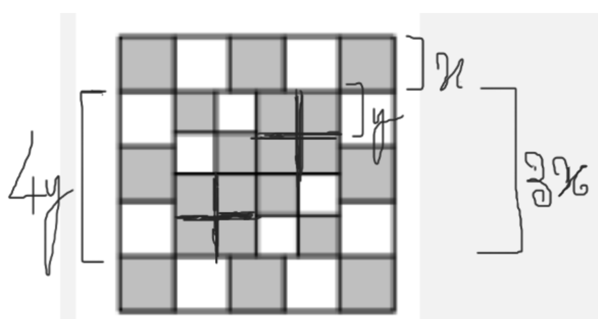
Raghav's idea was that the figure could be considered as a 5×5 square. There are 16 squares on the boundary, and only 8 of them are shaded. So exactly $\frac{1}{2}$ of the boundary squares is shaded.

Then, in the inner portion, we can divide them in a way and get 16 squares. Then exactly 12 of these smaller squares are shaded. So, exactly $\frac{3}{4}$ of the inner squares is shaded.



After this, I asked him what to do next. He was thinking at that time and couldn't find a good way to proceed further.

Here, I suggested him to take variables for the sides of the different squares that he could see, say let the side of the smaller square be x, and that of the larger one be y. He agreed to me.
Then, he almost immediately figured out an useful expression and replied that $3x = 4y$.
This was absolutely correct, and it comes since the sides of 4 smaller squares coincide perfectly with the sides of 3 larger squares:



Now, it was pretty simple. The total area of the figure was just $(16x^2+16y^2)$, and the shaded area was $(8x^2 + 12y^2)$. So we were looking for the value of the ratio:

$$\left(\frac{8x^2+12y^2}{16x^2+16y^2}\right)$$

At this point, Raghav told me of changing the denominator of $(16x^2+16y^2)$ directly to $25x^2$ (since from his point of view, $25x^2$ was just the area of the full square), but it wasn't necessary, because it directly came from the relation $3x = 4y$, that we had found earlier.

So instead, I took $y = \frac{3x}{4}$ and substituted it back into the expression:

$$\left(\frac{8x^2+12y^2}{16x^2+16y^2}\right) = \left(\frac{8x^2+12\left(\frac{9x^2}{16}\right)}{16x^2+16\left(\frac{9x^2}{16}\right)}\right)$$

$$\implies \left(\frac{\frac{128x^2+108x^2}{16}}{\frac{256x^2+144x^2}{16}}\right)$$

$$\implies \frac{236x^2}{400x^2}$$

$$\implies \frac{59}{100}.$$

So the answer was coming 59%. Raghav was still getting 68% somehow as his answer, so I double-checked myself and gave him some more time to check his answer again. And the answer was indeed 59%.

3 Question 2

We next moved to an easy problem.
The result of the calculation $(9 \times 11 \times 13 \times 15 \times 17)$ is the 6 - digit number $\overline{3n8185}$. What is the value of n?

I gave Raghav 5 minutes to solve this, and as expected, he quickly got the answer.

He just used the divisibility of 9 rule, and as 9 divides this number, 9 should also divide the sum of digits of the number. Hence, $9 \mid (25+n)$ and with $0 \leq n \leq 9$, the only possible value is $n = 2$. This was a very short problem.

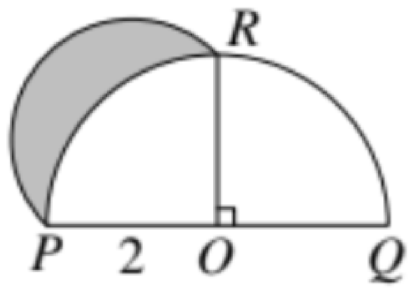
4 Question 3

Next we moved to a slightly harder but still an easy problem.
The positive integers m and n are such that $10 \times 2^m = 2^n + 2^{n+2}$. What is the difference between m and n?
Raghav took some time and proceeded in the correct way. He took 2^n common from the R.H.S. and this changes to :-
$\implies 10 \times 2^m = 2^n(1+4) = 2^n(5)$.

And from here it's very obvious that a factor of 5 can be cancelled both sides which gives :-
$\implies 2 \times 2^m = 2^{m+1} = 2^n$.
$\implies (m+1) = n$.
$\implies (m-n) = -1$.
And this is our required answer.

5 Question 4

Next we moved on to a moderate problem.
The diagram below, shows a semicircle with centre O and radius 2 and a semi-circular arc with diameter PR. Given $\angle POR = 90°$, find the area of the shaded region.



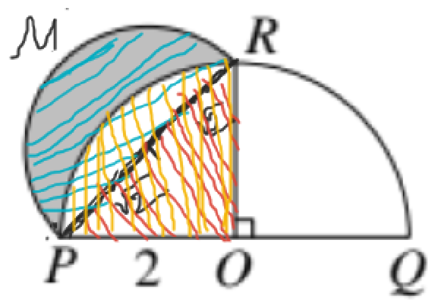
To start with, I joined PR and labelled some of the lengths. Raghav too quickly found $PR = 2\sqrt{2}$ from the Pythagoras Theorem.
Since the semi-circular arc is actually a semi-circle with diameter PR, it's radius would simply be $\sqrt{2}$.
Now, Raghav claimed that we should subtract the area of the triangle, from the area of the semi-circle of radius $\sqrt{2}$. This actually dosen't work, and he quickly realized it.
Then he correctly stated that we should add the areas of the both the semi-circle and the triangle, then subtract the area of a quarter circle.

This works, because (Area of blue) + (Area of red) - (Area of yellow) = (Area of shaded region). (See Figure in the next Page)
Thus, our required answer is :-

$$\frac{1}{2}\pi(\sqrt{2})^2 + \frac{1}{2}(2)(2) - \frac{1}{4}\pi(2^2)$$

$$\implies \pi + 2 - \pi = 2.$$

Raghav was making some mistake in his calculations and was getting $(\pi + 2)$ as



his answer, but he he quickly got it right.
Another interesting fact related to this problem, was that the curved shaded region is the smallest lune (as a shape) that actually equals the area of a polygon (in this case, the triangle). There has been a numberphile video on this. It can be checked here: Lunes - Numberphile .

6 Question 5

This was the final problem from our paper.
Sam writes on a white board the positive integers from 1 to 6 inclusive, once each. She then writes p additional fives and q sevens on the board. The mean of all the numbers on the board is then 5.3. What is the smallest possible value of q?

After giving Raghav some time, he said that the sum of numbers from 1 to 6 was just $\frac{(6)(7)}{2} = 21$, then we add a total of p more fives and q more sevens. So the total sum adds up to $(21 + 5p + 7q)$.
And the total number of numbers was $(6 + p + q)$.
Thus we had the relation:

$$\frac{(21 + 5p + 7q)}{(6 + p + q)} = 5.3$$

$$\implies (21 + 5p + 7q) = (5.3)(6 + p + q)$$

Raghav suggested to multiply by 10 both sides to get rid of the decimal point, so I did that:

$$\implies (210 + 50p + 70q) = 53(6 + p + q)$$

$$\implies (210 + 50p + 70q) = 318 + 53p + 53q$$

$$\implies 17q = 3p + 108$$

After here, Raghav said to take a factor of 3 common from the R.H.S to give: $17q = 3(p + 36)$.
Then he argued that since q is an integer, we can claim that $3 \mid 17q$, or basically $3 \mid q$. And this was correct.
And then we could basically check out all multiples of 3 in the value of q, from $(q = 3, 6, 9, \dots)$, till we get a positive integer solution for p.
Raghav did till here, and then in the end, found the smallest value of q to be 9, which is the correct answer.
Exercise:- Could you reason out why $q \leq 3, 6$ fails? Why does $q = 9$ work?

After this, we had completed solving all the 5 problems that we had been provided by cheenta. Since we had some more time left, I decided to continue the discussion by giving some of my own made problems.

7 Question 6

This was a moderate but (I hope) a nice problem.
Let A, B, C, D be distinct digits from 0 to 9 (A, B, C are non-zero), such that:

$$\begin{array}{cccc} & A & B & A \\ \times & & & B \\ \hline C & C & D & D \end{array}$$

Given this has a unique solution, find it.
Raghav started the problem by converting the column wise expression into an equation:

$$(101A + 10B)B = (1100C + 11D)$$

Then he noticed a factor of 11 on the R.H.S., so he took a factor of 11 there:

$$(101A + 10B)B = 11(100C + D)$$

Then exactly like we had done in the previous problem, where we could claim that $3 \mid 17q$, since here too we have A, B to be positive integers, we can claim that $11 \mid (101A + B)B$.
But, as B is a single digit number, $11 \nmid B$. Thus, we have that $11 \mid (101A+10B)$.
Then, Raghav immediately started using modulo 11. (Which was the best approach here).
We have that $101A \equiv 2A \pmod{11}$, because $101 = (11)(9) + 2$.
Similarly, we could say that $10B \equiv -B \pmod{11}$.
Thus, we can say that $11 \mid (2A - B)$.
At this point, Raghav said that $(A, B) = (6, 1)$ is a solution. He was partly right, because this set may not necessarily give solutions for C and D.

Moreover, as $11 \mid (2A - B)$, $(2A - B)$ can be any multiple of 11, starting from $(0, 11, 22, \dots)$. But as $0 < A, B \leq 9$, the only possible options are:
(i) $(2A - B) = 0$
(ii) $(2A - B) = 11$
For the first case, Raghav claimed that $A < 5$ and $A > 0$. Or basically, $0 < A < 5$ and there are 4 choices for A. Checking all those choices gives us 4 solutions, $(A, B) = (1, 2), (2, 4), (3, 6), (4, 8)$.
Now these are only valid solutions for (A, B), we had to put them back in the equation to check if it provides valid solutions for C and D.
A quick checking gives us that none of these are valid solutions. So we move to the second case.
To get $(2A - B) = 11$, Raghav quickly listed the valid cases, which are $(A, B) = (6, 1), (7, 3), (8, 5), (9, 7)$.
Again a quick checking gives us that $(A, B) = (7, 3)$ indeed works, and this is the solution with valid values for C and D:

$$\begin{array}{cccc} & 7 & 3 & 7 \\ \times & & & 3 \\ \hline 2 & 2 & 1 & 1 \end{array}$$

Since there only existed a unique solution (as I had made the problem, I had checked all cases before and only this gave a solution, so I mentioned specifically the solution is unique), we stopped here and moved on to the next problem.

Exercise:- Since Raghav chose to attempt this problem as an equation form, he missed a very easier approach which could have avoided the use of mod 11. Could you find it? (**Hint**: Divisibility Rule of 11)

8 Question 7

Next I gave him an easier problem but with a little twist at the ending.
Consider a polygon whose smallest angle is $30°$, and all the angles are consecutive multiples of $30°$. What is the maximum number of sides this polygon can have?
Since we were almost out of time, I decided to give the main idea to start.
At this point Raghav had already figured out that $n = 3$ works (for the case of a right triangle), but since I asked for maximum, he started thinking again.
Let the number of possible sides of the polygon be n.
So this polygon has its angles in the form $(30°, 60°, \cdots, 30n°)$.
Now, we add the angles. There is a well-known formula for the sum of angles of a polygon, which is $180(n - 2)$. (**Exercise**: Can you prove it?)
Thus, we had:

$$30 + 60 + \cdots + 30n = 180(n - 2)$$

We took a factor of 30 common both sides, and cancelled them, which gave us:

$$1 + 2 + \cdots n = 6(n - 2)$$

Raghav was already aware of the well-known formula $(1+2+\cdots+n) = \frac{n(n+1)}{2}$ (this was also used once in the 5th problem) (**Exercise**: Can you prove it?), so we used it here again:

$$\frac{n(n+1)}{2} = 6(n-2)$$

.

$$\implies n^2 + n = 12(n-2)$$

.

$$\implies n^2 - 11n + 24 = 0$$

.

Now a small factorization produced 2 roots, which are the only probable solutions to n. (He was also aware that $n = 3$ works, and quickly found the other root)

$$\implies (n-3)(n-8) = 0$$

Thus the roots are $3, 8$.

Raghav quickly claimed that the maximum number of possible sides to the polygon should be 8. But much to his surprise, and as it really should have been, the answer is not 8, but 3 only.

To find the reason, I gave Raghav some time to think. He thought that reflex angles occur for $n = 8$, which he is of course right, but they do not create much of a problem, as I reminded him that concave polygons too (those that have reflex angles) can exist. The problem was into something else.

Although I told Raghav the reason because the class had almost ended (he did say the idea was pretty interesting and it had a different way of thinking), I will not disclose the reason here, and let you all, the readers, try finding it.

Exercise:- Can you find the reason why $n = 8$ fails as a solution?

So that was all which we had discussed in the 10th Session. We had discussed 5 problems which we had in the problem set, and 2 additional problems that I had made myself. And, I have tried to make this session as interesting, informative, and as enjoyable as possible.

Thank You For Reading!!