$\cos 2x = \cos^2 x - \sin^2 x.$

Group A

1. Define $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \begin{cases} \cos(2x) & \text{if } x \text{ is rational,} \\ \sin^2(x) & \text{if } x \text{ is irrational.} \end{cases}$$

Find all the real numbers where

(a) f is continuous,

(b) f is differentiable.

$E(X|Y=1)$

2. Consider the matrix

$$\mathbf{P} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \\ 8 & 5 \end{pmatrix}_{3\times 2}.$$

Find a matrix $\mathbf{G}$ such that $\mathbf{AGA} = \mathbf{A}$ where $\mathbf{A} = \mathbf{PP}^{\mathrm{T}}$.

3. Consider the set of all five digit integers formed by permuting the digits 1, 2, 4, 6 and 7. Let X denote a randomly chosen integer from this set and let Y denote the position, from the right, of the digit 4 in the randomly chosen integer. For example, if the integer chosen is 12647, then Y is 2 and for 41276, Y is 5.

(a) Find $E(X)$ and $E(Y)$.

(b) Find $E[X|Y = y]$ for all possible values of y.

(c) Show that X and Y are uncorrelated but not independent.

$\cos^2 x - \sin^2 x$

$P(X|Y=y_1)$

$P(A\cap B) + P(A\cap B^c) = P(A)$

Group B

4. Let U_1, U_2, U_3 be i.i.d. random variables which are uniformly distributed on $(0,1)$. Let $X = \min(U_1, U_2)$ and $Y = \max(U_2, U_3)$.

   (a) Find $P(X \leq x, Y \leq y)$ for all $x, y \in \mathbb{R}$.

   (b) Find $P(X = Y)$.

   (c) Find $E[X I_{\{X=Y\}}]$ where I_A is the indicator function of A.

5. Consider a game with six states $1, 2, 3, 4, 5, 6$. Initially a player starts either in state 1 or in state 6. At each step the player jumps from one state to another as per the following rules.

   A perfectly balanced die is tossed at each step.

   (i) When the player is in state 1 or 6: If the roll of the die results in k then the player moves to state k, for $k = 1, \ldots, 6$.

   (ii) When the player is in state 2 or 3: If the roll of the die results in 1, 2 or 3 then the player moves to state 4. Otherwise the player moves to state 5.

   (iii) When the player is in state 4 or 5: If the roll of the die results in 4, 5 or 6 then the player moves to state 2. Otherwise the player moves to state 3.

The player wins when s/he visits 2 more states, besides the starting one.

(a) Calculate the probability that the player will eventually move out of states 1 and 6.

(b) Calculate the expected time the player will remain within states 1 and 6.

(c) Calculate the expected time for a player to win, i.e., to visit 2 more states, besides the starting one.

6. Let $X_1, \ldots, X_n$ be i.i.d. random variables which are uniformly distributed on $(\theta, 2\theta)$, $\theta > 0$.

(a) Show that $\frac{X_{(n)}}{2}$ is the maximum likelihood estimator (MLE) of θ where $X_{(n)} = \max(X_1, \ldots, X_n)$.

(b) Find an unbiased estimator for θ based on the MLE.

(c) Given any $\epsilon > 0$, show that

$$\lim_{n \to \infty} P\left(\left| \frac{X_{(n)}}{2} - \theta \right| > \epsilon \right) = 0.$$

7. There are two urns each contains N balls numbered from 1 to N. From each urn a sample of size n is selected without replacement. Denote the set of numbers appearing in the first and second samples by $s_1 = \{i_1, \ldots, i_n\}$ and $s_2 = \{j_1, \ldots, j_n\}$ respectively. Let

$$X = |s_1 \cap s_2| = \text{number of common elements in } s_1 \text{ and } s_2.$$

(a) Find the probability distribution of X.

(b) Suppose $n = 6$ and the observed value of X is 4. Obtain a method of moments estimate of N.

8. Let X_1, X_2, X_3 be i.i.d. random variables from $N(\mu, \sigma^2)$. Let

$$\bar{X} = \frac{1}{3}\sum_{i=1}^{3} X_i, \quad T_1 = \sum_{i=1}^{3} X_i^2, \quad T_2 = \frac{1}{3}\sum_{i=1}^{3}(X_i - \bar{X})^2.$$

(a) Compute $E[T_1|\bar{X}]$ and $E[T_1|T_2]$

(b) Obtain the exact critical region of a level α $(0 < \alpha < 1)$ test for $H_0 : \mu = 0$ vs $H_1 : \mu \neq 0$ that rejects H_0 if and only if $\frac{\bar{X}^2}{T_1}$ is sufficiently large.

9. Consider a linear regression model:

$$y_i = \alpha + \beta x_i + e_i, \; i = 1, 2, \ldots, n$$

where x_i's are fixed and e_i's are i.i.d. random errors with mean 0 and variance σ^2.

Define two estimators of β as follows

$$\widehat{\beta}_1 = \frac{\sum_{i=1}^n y_i}{\sum_{i=1}^n x_i} \quad \text{and} \quad \widehat{\beta}_2 = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}.$$

(a) Obtain an unbiased estimator of β as a linear combination of $\widehat{\beta}_1$ and $\widehat{\beta}_2$.

(b) Find mean squared errors of $\widehat{\beta}_1$ and $\widehat{\beta}_2$. Which, between $\widehat{\beta}_1$ and $\widehat{\beta}_2$, has lower mean squared error?