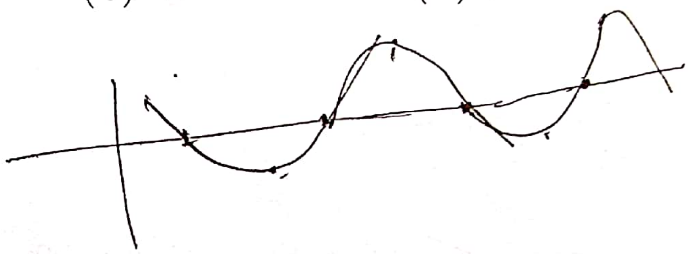
1. Let f be a differentiable function on $\mathbb{R}$ such that the number of elements in the set $\{x : f(x) = 0\}$ is 50. Then the minimum number of elements in the set $\{x : f'(x) = 0\}$ is

(A) 49 (B) 50 (C) 51 (D) 0



2. Let $\mathcal{P}_3$ be the vector space of all polynomials with real coefficients and of degree less than or equal to 3 over the field of real numbers. The dimension for the subspace of polynomials $p(x)$ such that $p(1) = p(2) = 0$ is

(A) 0 (B) 2 (C) 1 (D) 3

$(x-1)(x-2)$, $x(x-1)(x-2)$

$\{ (x-1)(x-2), x(x-1)(x$

3. Let k denote the number of 2×2 real matrices X with

$$X^2 = \begin{bmatrix} 10 & 9 \\ 4 & 10 \end{bmatrix}.$$

Then

(A) $k = 0$ (B) $k = 3$ (C) $k = 2$ (D) $k = 1$

$a_1^2 + a_2 a_3 = 10$

$a_4^2 + a_2 a_3 = 10$

$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$

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$$\frac{\frac{\sin x^4}{x^4}\cdot x^4}{\left(\frac{\sin x}{x}\right)^3 \cdot x^3} \qquad 1 \qquad = 0$$

4. $\lim_{x\to 0} \dfrac{\sin x^4}{(\sin x)^3}$ is

(A) 1 (B) 0 (C) ∞ (D) $\frac{4}{3}$

5. Let p and q be two non-zero real numbers and $pq \neq -1$. Suppose the roots of the equation

$$\left(p+\frac{1}{q}\right)x^2 - \left(p+\frac{1}{p}+q+\frac{1}{q}\right)x + \left(q+\frac{1}{p}\right) = 0$$

are distinct and rational. Then

$\Delta > 0$

(A) $p = q$

(B) p and q MUST be rational numbers

(C) $\frac{p}{q}$ MUST be a rational number

(D) $p+q$ MUST be a rational number

$\left(p+\frac{1}{p}+q+\frac{1}{q}\right)^2 - 4\left(p+\frac{1}{q}\right)\left(q+\frac{1}{p}\right)$

$(a+b)^2 - 4ab > 0$

$= (a-b)^2$

$a-b$

$p+\frac{1}{q}-q-\frac{1}{p} \neq 0$ $\quad p+\frac{1}{q}$

6. Let

$$\alpha = \lim_{x\to\infty} \frac{[x^3]}{x^3} \quad \text{and} \quad \beta = \lim_{x\to\infty} \frac{[x]^3}{x^3},$$

$n^3-1 < n^3 < n^3$

$1-\frac{1}{n}$

where $[x]$ denotes the greatest integer less than or equal to x. Then

(A) $\alpha = 1, \beta = 1$

(B) $\alpha = 1, \beta = 0$

(C) $\alpha = 1$, β does not exist

(D) Neither α nor β exists

$n-1 < [x] < n$

$(n-1)^3 < [x]^3 < n^3$

$\left(1-\frac{1}{n}\right)^3$

$p+\frac{1}{p}+q+\frac{1}{q}$ $\quad -q+\frac{1}{q}-\frac{1}{p}$

$2q+\frac{1}{2p}$

7. For every natural number n, let

$$I_n = \int_0^n (\{x\}^n + \{-x\}^n)\, dx$$

where $\{x\} = x - [x]$ where $[x]$ denotes the greatest integer less than or equal to x. Then I_n equals

(A) 0 (B) $\frac{2n^{n+1}}{n+1}$ (C) $\frac{2n}{n+1}$ (D) n

8. The lengths of the sides of a right-angled triangle are numerically equal to the arithmetic mean, the geometric mean and the harmonic mean of two distinct positive real numbers a, b, $a > b$. Then $\frac{a}{b}$ is

(A) $5+\sqrt{2}$

(B) $2+\sqrt{5}$

(C) $2+\sqrt{3}$

(D) $3+\sqrt{2}$

9. The number of non-real roots of the polynomial equation $x^{12} + 2x^6 = 5$ is

(A) 0 (B) 2 (C) 6 (D) 10

~~10~~. A matrix is chosen at random from the set of all 2×2 matrices with elements 0 or 1 only. The probability that the determinant of the chosen matrix is non-zero is

(A) $\frac{3}{16}$ (B) $\frac{3}{8}$ (C) $\frac{1}{4}$ (D) $\frac{1}{8}$

$\frac{6}{16} = \frac{1}{6}$ $\frac{3}{8}$. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \Rightarrow ad-bc \neq 0$. (i) $a=0$, $a=1, d=1$ (ii) 3

~~11~~. If $({}^nC_0 + {}^nC_1)({}^nC_1 + {}^nC_2)\ldots({}^nC_{n-1} + {}^nC_n) = k\,{}^nC_0\,{}^nC_1\ldots{}^nC_{n-1}$, then k is equal to

(A) $\frac{(n+1)^n}{n!}$

(B) $\frac{n^n}{n!}$

(C) $\frac{(n+1)^n}{nn!}$

(D) $\frac{(n+1)^{n+1}}{n!}$

${}^nC_0 + {}^nC_1 = \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

${}^nC_k + {}^nC_{k-1} = \binom{n+1}{k}$

$\frac{\binom{n+1}{k}}{\binom{n}{k}} = \frac{(n+1)!}{(n+1-k)!\,k!} \cdot \frac{(n-k)!\,k!}{n!}$

$(n+1)\cdot\frac{(n-k)!}{(n-k+1)!} = \frac{(n+1)^n}{n+1-k}$. $\frac{(n+1)^n}{n!}$

12. Let $\pi = (a_1, a_2, \cdots, a_{2021})$ be a permutation of $(1, 2, \cdots, 2021)$. For every such permutation π,

$$P(\pi) = \prod_{j=1}^{2021} (a_j - j).$$

Then

(A) $P(\pi) = 0$ for all π

(B) $P(\pi)$ is even for all π

(C) $P(\pi)$ is odd for all π

(D) There exists π_1 and π_2 for which $P(\pi_1)$ is even and $P(\pi_2)$ is odd.

$C^K = \begin{bmatrix} 4^K & 0 \\ 0 & 1 \end{bmatrix}$. $\quad B^K = \begin{bmatrix} 2^K & 0 \\ 0 & 3^K \end{bmatrix}$.

13. Let $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$. Then the sum

$$\operatorname{tr} A + \operatorname{tr} \frac{BAC}{9} + \operatorname{tr} \frac{C^2 AB^2}{9^2} + \cdots + \operatorname{tr} \frac{B^{2n-1} AC^{2n-1}}{9^{2n-1}} + \operatorname{tr} \frac{C^{2n} AB^{2n}}{9^{2n}} + \cdots$$

(A) is equal to 12

(B) is equal to 14

(C) is equal to 9

(D) is ∞

$\therefore C^K A B^K$

$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$

6 + 5 ... 2 ... K ... 6

14. Let $\mathcal{F}$ be the set of all bijections f from the set $S = \{1, 2, 3, 4, 5, 6\}$ onto itself such that $f(f(i)) \neq i$ for any $i \in S$. The number of elements in $\mathcal{F}$ are

(A) 265 (B) 120 (C) 200 (D) 160

15. Let $f : [0, 1] \to \mathbb{R}$ be a monotonic function such that the number of its discontinuity points is finite. For $t \geq 0$ let $m_t = \int_0^1 x^t f(x) dx$. Then which of the following statements is true?

(A) There exists a $t \geq 0$ for which m_t is not defined. ✗

(B) m_t is defined and is equal to ∞ for all $t \geq 0$. ✗

(C) m_t is defined for all $t \geq 0$ and there exists $t_1, t_2 \geq 0$ such that m_{t_1} is equal to ∞ and $m_{t_2} < \infty$. ✗

(D) $m_t < \infty$ for all $t \geq 0$.

$\frac{1}{t+2}$

$(a_1 - 1)(a_2 - 2)(a_3 - 3)$.

$\int_0^2 \{x\}\, dx = \int_0^1 x - \lfloor x \rfloor\, dx + \int_1^2 x - \lfloor x \rfloor\, dx$

$= \frac{x^2}{2} + \int_1^2 (x-1)\, dx = \frac{x^2}{2} - x \Big|_1^2$

~~16~~. Let $F(x)$ and $G(x)$ be cumulative distribution functions of random variables X and Y, respectively. Then which of the following would NOT be a cumulative distribution function?

(A) $H(x) = (F(x))^2$ ✓

(B) $H(x) = (F(x) + G(x))/2$ ✓

(C) $H(x) = (1 - F(-x) + F(x))/2$

(D) $H(x) = \sqrt{G(x)}$ ✓

$1 - P(X \le -x)$

$\Rightarrow P(X > -x) + P(X < x)$

~~17~~. Suppose that X and Y are independent and identically distributed exponential random variables with mean $1/\lambda$ where $\lambda > 0$. Then $P(X > 4Y | X < 5Y)$ equals

(A) $4/5$

~~(B)~~ $1/25$

(C) $e^{-5\lambda}/e^{-4\lambda}$

(D) $\left(e^{-4\lambda} - e^{-5\lambda}\right) / \left(1 - e^{-4\lambda}\right)$

$\frac{P(4Y < X < 5Y)}{P(X < 5Y)}$

$P(X \le kY) = \int_{y=0}^{\infty} \int_0^{ky} \lambda e^{-\lambda x}\, dx$

~~18~~. Suppose X_1 and X_2 are independent and identically distributed exponential random variables with mean 2. The expectation of $\max\{X_1, X_2\}$ is

(A) 3 (B) 4 (C) 3.5 (D) 2.5

19. Let X be a random variable. Let $F(t)$ denote its cumulative distribution function and $M(t)$ denote its moment generating function. Consider the following two statements.

$P(X \le t)$
$= 1 - P(X \le -t)$
$= P(X \ge -t)$

(I) $F(t) = 1 - F(-t)$ for $t \in \mathbb{R}$

(II) $M(t) = M(-t)$ for $t \in \mathbb{R}$

Which of the above statements are individually equivalent to random variable X being symmetric about zero?

(A) Neither (I) nor (II)

(B) (I) and (II)

(C) (I) but not (II)

(D) (II) but not (I)

$E(e^{tX}) = E(e^{-tX})$

$P(X \le t)$
$= P(-X \le t)$

20. Suppose that $X_1, X_2, \ldots$ are independent and identically distributed exponential random variables with mean $\lambda > 0$. For $n \ge 1$, define $Y_n = X_1^2 + X_2^2 + \cdots + X_n^2$. Then

$$\lim_{n \to \infty} \frac{V(Y_n)}{n} =$$

(A) $20\lambda^4$

(B) $2\lambda^2(2\lambda^2 - 1)$

(C) $2\lambda^2$

(D) λ^4

$Var(X_i^2) = E(X_i^4) - E(X_i^2)^2$

$\frac{10\bar{x}+y_1+y_2}{n+2} = 10$. $\frac{x_1+x_2+\dots+x_n}{n} = 10$. $\frac{x_1+\dots+x_n+y_1+y_2}{n+2} = 10$

~~21~~. The mean and the variance of a set of observations are 10 and 25 respectively. When two new observations are included in the set, both the mean and the variance remain the same. Which of the following is a correct statement?

(A) the two new observations are both 10

(B) the two new observations are 15 and 5

(C) the two new observations are 20 and 0

(D) the two new observations cannot be uniquely determined unless the number of observations are known

$10n + (y_1+y_2) = 10n + 20$, $y_1+y_2 = 20$. $\frac{x_1+x_2}{2} = 10 \Rightarrow x_1+x_2 = 20$. $\frac{x_1+x_2+y_1+y_2}{4} = 10 \Rightarrow x_1+x_2+y_1+y_2 = 40 \Rightarrow y_1+y_2 = 20$

T 22. Consider data on three variables X, Y and Z. Suppose least squares regressions of Y on X, Y on Z and X on Z are performed. If the slope coefficient in each of the three regressions is 2, what can be said about the slope coefficient of the least squares regression of Y on $(X+Z)$?

(A) It is greater than 2

(B) It is equal to 2

(C) It is less than 2

(D) It cannot be determined whether it is more than 2 or less than 2 but it cannot be equal to 2

A : detects infection
B : Diabetic .

$P(A \mid B) = 0.9$.
$P(A \mid B^c) = 0.8$.

23. Suppose that the probability of a diagnostic test detecting the presence of an infection is 0.9 for a diabetic individual and 0.8 for a non-diabetic individual. An individual is randomly selected from the population. What should be the ratio of the proportions of diabetic to non-diabetic individuals in the population so that the probability of the selected individual being diabetic given that the above diagnostic test produces a negative result is the same as that of the individual being non-diabetic given that the test produces a positive result?

(A) 2 : 1 (B) 3 : 2 (C) 5 : 3 (D) 4 : 3

$P(B \mid A^c) = P(B^c \mid A)$.

$= \dfrac{P(A \mid B^c)\,P(B^c)}{P(A \mid B^c)\,P(B^c) + P(A \mid B)\,P(B)}$

2) $\dfrac{P(A^c \mid B).\,P(B)}{P(A^c \mid B)P(B) + P(A^c \mid B^c)\,P(B^c)}$.

$0.9\,P($

$0.1\,P(B) + 0.2\,(1 - P(B))$

$0.2 - 0.2$

$\dfrac{(0.1)\,P(B)}{0.2 - 0.1\,P(B)}$

$= (0.8)(1-$

× 24. Consider independent random samples $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ $(m, n \geq 2)$ from $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$, respectively. Suppose we want to test $H_0 : \mu_1 = \mu_2$ vs $H_1 : \mu_1 \neq \mu_2$ at level of significance α $(0 < \alpha < 1)$. Then

(A) the two sample t-test statistic yields more power than the ANOVA statistic for any non-zero value of $\mu_1 - \mu_2$ and for any m, n.

(B) the two sample t-test statistic yields less power than the ANOVA statistic for any non-zero value of $\mu_1 - \mu_2$ and for any m, n.

(C) the two sample t-test statistic and the ANOVA statistic yield the same power for any non-zero value of $\mu_1 - \mu_2$ and for any m, n.

(D) the two sample t-test statistic and the ANOVA statistic will yield the same power for any non-zero value of $\mu_1 - \mu_2$ if $m = n$ but will not necessarily yield the same power if $m \neq n$.

† 25. In order to compare between $t(\geq 2)$ treatments, one carries out t replications of an experiment using a randomized block design with t blocks. The error degrees of freedom for testing the equality of treatment effects in such a design is

(A) $t + 1$

(B) $3(t - 1)$

(C) $t(t - 1)(t + 1)$

(D) $(t - 1)(t^2 + t - 1)$

26. Let X_i be independently distributed Poisson$(i\lambda)$ random variables, $i = 1, \ldots, n$ and $\lambda > 0$, is an unknown parameter. Assume $\sum_{i=1}^{n} X_i > 0$. Then the MLE of λ is

(A) $\dfrac{\sum_{i=1}^{n} iX_i}{(n+1)}$

(B) $\dfrac{2\sum_{i=1}^{n} X_i}{n(n+1)}$

(C) $\dfrac{\sum_{i=1}^{n} X_i}{n(n+1)}$

(D) $\dfrac{\sum_{i=1}^{n} iX_i}{n}$

$+ \sim P\left(\frac{n(n+1)}{2}\lambda\right)$

Assume $x > 0$

$1 - P(X_i = 0)$

$f(x_i) = \dfrac{e^{-\lambda i}(\lambda i)^{x_i}}{(x_i)!}$

$\dfrac{e^{-\sum \lambda i}(\lambda 1)^{x_1}(\lambda 2)^{x_2}\cdots}{\prod x_i!}$

$Y_1 \sim N(0, 2\sigma^2)$

$Y_2 \sim N(0, 2\sigma^2)$

27. Suppose X_1, X_2, X_3 are independent normally distributed random variables with mean μ and variance σ^2. However, instead of X_1, X_2, X_3, we only observe $Y_1 = X_2 - X_1$ and $Y_2 = X_3 - X_2$. Which of the following statistics is sufficient for σ^2?

(A) $Y_1^2 + Y_2^2 + Y_1 Y_2$

(B) $Y_1^2 + Y_2^2 - Y_1 Y_2$

(C) $Y_1^2 + Y_2^2 + 2Y_1 Y_2$

(D) $Y_1^2 + Y_2^2$

$f(\underset{\sim}{x}) = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-\frac{1}{2\sigma^2}\sum (x_i-\mu)^2}$

$Y_1^2 = X_2^2 + X_1^2 - 2X_1X_2$

$Y_2^2 = X_3^2 + X_2^2 - 2X_2X_3$

$Y_1Y_2 = (X_2 - X_1)(X_3 - X_2)$

28. Consider data on body mass index (BMI) and fasting blood sugar levels (fgl) for two groups of individuals. Suppose that the correlation coefficient between BMI and fgl is equal to 0.5 in each of the two groups. If the two groups are combined, the correlation coefficient between BMI and fgl:

(A) will be equal to 0.5

(B) will be positive but not necessarily be equal to 0.5

(C) can be negative

(D) can be zero but cannot be negative

$f(y_1, y_2) = \frac{1}{\sqrt{2\pi}\sqrt{2}\sigma} e^{-\frac{1}{2}\cdot\frac{y_1^2}{2\sigma^2}}$

$\therefore f(y_1, y_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}} \sum x_i$

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$P(Z = z) = P(\cancel{X = x \mid Y})$ $\int P(Z = z \mid \cancel{X = x}\ Y = x)$

29. A random variable Z is obtained as follows. Let $X \sim U(0,1)$, and Y given $X = x$ be Bernoulli with probability of success x. If $Y = 1$, Z is defined to be X. Otherwise, the experiment is repeated until a pair (X, Y) with $Y = 1$ is obtained. Then, the probability density function of Z on $(0,1)$ is

$X \sim \mathrm{Uni}(0,1)$
$Y \mid X = x \sim \mathrm{Ber}(x)$

(A) $2z$

(B) $2(1-z)$

(C) $12z^2(1-z)$

(D) $6z(1-z)$

$Z = \{X, Y = 1$

$T = K$

$Z = \{X, \text{w.p.}$
$0, 0, 0 \ldots K.$

$P(T = K) = \frac{1}{2^K}$

$P(Y=1) = \int_0^1 P(Y = 1 \mid X = x) \cdot f(x)\,dx$

$\int_0^1 x\,dx = \frac{1}{2}$

$(Z = 3 \mid T$

x

30. Suppose $X_1, X_2, \ldots, X_9$ are independent variables such that X_i is distributed as Poisson with mean i, $i = 1, 2 \ldots, 9$.

The conditional variance of $\sum_{i=1}^{5} X_i$ given $\sum_{i=1}^{9} X_i = 60$ is

(A) 20 (B) 40/3 (C) 100/3 (D) 400/27

$\sum X_i$ let $Y_1 = \sum_{i=1}^{5} X_i$, $Y_2 = \sum_{i=6}^{9} X_i$ independent

$Y_1 \sim \mathrm{Poi}(15)$

$Y_2 \sim \mathrm{Poi}(30)$

$Y_1 \mid Y_1 + Y_2 = 60$

$\mathrm{Nm}(X \mid Z)$